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Anosov maps with rectangular holes. Nonergodic cases.

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— *To the memory of Ricardo Mañé*

Abstract. We study Anosov diffeomorphisms on manifolds in which some 'holes' are cut. The points that are mapped into our holes will disappear and never return. We study the case where the holes are rectangles of a Markov partition. Such maps with holes generalize Smale's horseshoes and certain open billiards. The set of nonwandering points of our map is a Cantor-like set we call a *repeller*. In our previous paper, we assumed that the map restricted to the remaining rectangles of the Markov partition is topologically mixing. Under this assumption we constructed invariant and conditionally invariant measures on the sets of nonwandering points. Here we relax the mixing assumption and extend our results to nonmixing and nonergodic cases.

1. Introduction

Let $T : M' \to M'$ be a topologically transitive Anosov diffeomorphism of class $C^{1+\alpha}$ on a compact Riemannian manifold M'. Sinai [**12**] and Bowen [**1**] constructed Markov partitions for transitive Anosov diffeomorphisms. Let $\mathcal{R}'$ be an arbitrary Markov partition of M' into rectangles $R_1, \dots, R_{I'}$. We assume that these rectangles are small enough, so that the symbolic dynamics is well defined [**12,1**].

Let $I < I'$. Put $H = \cup_{i=I+1}^{I'} (int\, R_i)$ and $M = M' \setminus H$. Then M is a manifold with boundary. We study here the dynamics of T restricted to M, thinking of H as a 'hole' into which some points of M will be mapped by T, and then they disappear (escape).

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Notation. For any $n \geq 0$ we put

$$M_n = \cap_{i=0}^n T^i M \quad and \quad M_{-n} = \cap_{i=0}^n T^{-i} M,$$

and also

$$M_+ = \cap_{n \geq 1} M_n, \quad M_- = \cap_{n \geq 1} M_{-n}, \quad \Omega = M_+ \cap M_-$$

All these sets are closed, $T^{-1}M_+ \subset M_+$, $TM_- \subset M_-$ and $T\Omega = T^{-1}\Omega = \Omega$. The set Ω consists of nonwandering points, i.e. those which never escape through holes, either in the future or in the past. The sets M_+ and M_- consist of nonwandering points in the past and the future, respectively. The purpose of this paper is the study of the dynamics T on Ω, M_+ and M_-.

A pictorial model of this type of dynamics was proposed by Pianigiani and Yorke [**10**]. Imagine a Sinai billiard table (with dispersing boundary), so that the dynamics of the ball are strongly chaotic. Let one or more holes be cut in the table, so that the ball can fall through. In particular, one can place those holes at the corners of the table and make 'pockets'. Let the initial position of the ball be chosen at random with some smooth probability distribution (this may be the equilibrium distribution). Denote by $p(t)$ the probability that the ball stays on the table for at least time t and, if it does, by $\mu(t)$ its (normalized) distribution on the table at time t. Natural questions are: does $p(t)$ converge to zero at some exponential rate, as $t \to \infty$? is there a limit probability distribution $\mu_+ = \lim_{t\to\infty} \mu(t)$; is that limit distribution independent of the initial distribution $\mu(0)$? These questions still remain open.

Pianigiani and Yorke [**10**] introduced a simpler class of dynamical systems - expanding (noninvertible) maps with holes, for which the above questions were answered positively in Refs. [**10,5**]. The limit probability distribution μ_+ is called *conditionally invariant measure.* The measure μ_+ is not invariant under T, it cannot be because of the holes. Instead, its image under T is proportional to itself: $\mu_+(T^{-1}(A \cap M_1)) = \lambda_+ \mu_+(A)$ for any Borel $A \subset M$ with some constant $\lambda_+ \in (0,1)$, which we call the *eigenvalue* of μ_+, cf. [**4**].

In 1981-86 Čencova [**2,3**] studied a class of invertible transformations with holes, namely smooth Smale's horseshoes. She also answered the above questions positively. In addition, she studied an inverse limit of the iterations of the measure μ_+ (pulled backward in time). The resulting limit measure, η_+, is invariant under the dynamics and supported on the set Ω of nonwandering points. That set is a Cantor-like closed set, sometimes called a *repeller* or a *semi-attractor*.

In 1994, Collet, Martinez and Schmitt [**5**] constructed invariant measures on the sets of nonwandering points (repellers) for Pianigiani-Yorke noninvertible transformations. They proved that the measure η_+ is a Gibbs measure, and thus it enjoys good statistical properties.

An special example of invertible hyperbolic systems with holes other than horseshoes was studied by Lopes and Markarian in [**9**]. That was an open billiard system – a particle bouncing off three circular scatterers placed sufficiently far apart on an open plane. They constructed measures μ_+ and η_+ and showed that η_+ was a Gibbs measure, too.

In Ref. [**4**] we generalized the above classes of invertible transformations with holes. We studied $C^{1+\alpha}$ Anosov diffeomorphisms with 'rectangular' holes just as described above, under an additional 'mixing condition':

Mixing condition. The symbolic dynamics generated by the partition $\mathcal{R} = \{R_1, \dots, R_I\}$ of M is a topologically mixing subshift of finite type. Equivalently, there is a $k_0 \geq 1$ such that $\mathrm{int}R_i \cap T^{k_0}(R_j \cap M_{-k_0}) \neq \varnothing$ for all $i, j \leq I$.

This class covered both Smale's horseshoes and open billiard tables (in any dimensions). We proved the existence and uniqueness of the measures μ_+ and η_+. We showed that η_+ was a Gibbs measure and found its potential function and topological pressure. We found necessary and sufficient conditions under which the measure η_+ coincided with the measure η_- constructed in the same way for the inverse map T^{-1}. This last result was never discussed in [**2,3,9**]. In particular, we showed that $\eta_+ = \eta_-$ for open billiard tables answering a question posed in [**9**].

In this paper we relax the mixing condition, thus allowing multiple ergodic components, periodic structure of ergodic components and nonrecurrent states as well. We prove the existence of the measures μ_+ and ν_+ and discuss their uniqueness and other properties. One of the most remarkable results is that the eigenvalue λ_+ of the map T on M equals the largest of the eigenvalues of T restricted to its ergodic components components. The conditionally invariant measure μ_+ is determined by that of the component with the largest eigenvalue. The invariant measure ν_+ coincides with the one on the ergodic component with the largest eigenvalue, as if the others did not exist.

The importance of the present study is in the following possible construction. Let $T : M' \to M'$ be an Anosov diffeomorphism and $H \subset M'$ be an arbitrary hole with smooth boundary, not necessarily connected (this is a physically interesting model!). To study the dynamics of T on $M = M' \setminus H$ one can approximate the hole H by a union, $H^{(r)}$, of 'rectangular' holes taking, for a sufficiently fine Markov partition, all its rectangles intersecting H. The union $H^{(r)}$ of 'rectangular' holes can be made arbitrarily close to the original hole H, and then one can possibly approximate the measures μ_+, η_+ for the map T on M by the measures $\mu_+^{(r)}, \eta_+^{(r)}$ for the map T on $M^{(r)} = M' \setminus H^{(r)}$. The work in this direction is currently underway. However, the map T on $M^{(r)}$ most certainly fails to satisfy the above mixing condition, so we cannot use our previous results in Ref. [4] directly. We have to relax the mixing condition first, and here we do just that.

Section 2 contains necessary results from Ref. [4]. In Section 3 we establish new results (still under the mixing condition), which we will need further. In Section 4 we discuss the case where the subshift generated by $\mathcal{R}$ is topological transitive but not topologically mixing. In Section 5 we study nonrecurrent rectangles. In Section 6 we consider the coexistence of two transitive classes of rectangles. Section 7 covers the cases of three or more transitive classes. Section 8 contains general conclusions on arbitrary number of transitive classes.

2. Necessary results for the mixing case

Here we recall the results of [4] which we are going to extend to non-mixing cases.

Denote by

$$\mathcal{U}' = \vee_{n=0}^{\infty} T^n \mathcal{R}' \quad and \quad \mathcal{S}' = \vee_{n=0}^{\infty} T^{-n} \mathcal{R}'$$

the partitions of the rectangles $R \in \mathcal{R}'$ into unstable and stable manifolds (fibers), respectively. The restrictions of $\mathcal{U}'$ to M and M_+ are denoted by $\mathcal{U}$ and $\mathcal{U}_+$, respectively. Similarly, we have partitions $\mathcal{S}$ and $\mathcal{S}_-$ of the sets M and M_- into stable fibers. Atoms $U \in \mathcal{U}$ and $S \in \mathcal{S}$ of these partitions are closed domains on unstable and stable manifolds. For any $x \in M'$ denote by $J^u(x)$ and $J^s(x)$ the Jacobians of the map DT restricted to the unstable and unstable subspaces at x, respectively.

Fact. [**12,4**]. There is a unique family of probability measures ν_U^u on fibers $U \in \mathcal{U}'$ such that

(i) (smoothness) ν_U^u is absolutely continuous with respect to the Riemannian volume on U, and its density, $\rho_U^u(x)$, $x \in U$, is Hölder continuous (see a convention below);

(ii) (conditional invariance) for any $x \in U_1 \in \mathcal{U}'$ and $Tx \in U_2 \in \mathcal{U}'$ we have

$$\rho_{U_1}^u(x) = \nu_{U_1}^u(T^{-1}U_2) \cdot J^u(x) \cdot \rho_{U_2}^u(Tx) \tag{2.1}$$

Equivalently, if $TU = U_1 \cup \cdots \cup U_L$, where $U_i \in \mathcal{U}$, then

$$\nu_U^u(U \cap T^{-1}A) = \sum_{i=1}^{L} \nu_U^u(T^{-1}U_i) \cdot \nu_{U_i}^u(A \cap U_i) \tag{2.2}$$

for any Borel set $A \subset M'$. The densities $\rho_U^u(x)$ satisfy the equation [**12**]

$$\frac{\rho_U^u(x)}{\rho_U^u(y)} = \lim_{n\to\infty} \frac{J^u(T^{-n}y)\cdots J^u(T^{-1}y)}{J^u(T^{-n}x)\cdots J^u(T^{-1}x)} \tag{2.3}$$

for all $x, y \in U$.

Convention. [4]. All the densities of measures on unstable and stable fibers are assumed to be Hölder continuous with the same Hölder exponent α, as the one of the derivative of the map T. We call a measure μ on M smooth if its conditional measures on unstable fibers $U \in \mathcal{U}$ are absolutely continuous with Hölder continuous densities.

Recall that every transitive Anosov diffeomorphism has a unique Sinai-Bowen-Ruelle [**13,1,11**] measure (SBR-measure), whose conditional distributions on unstable manifolds $U \in \mathcal{U}'$ are exactly our ν_U^u. Motivated by that, we will call ν_U^u u-SBR measures (on unstable manifolds).

For any Borel measure μ on M we define its norm by $||\mu|| = \mu(M)$. We denote by T_* the adjoint operator on the class of Borel measures on M defined by

$$(T_*\mu)(A) = \mu(T^{-1}(A \cap M_1))$$

for any $A \subset M$. Due to the holes, the operator T_* does not preserve norm. We denote by T_+ the (nonlinear) transformation on the set of probability measures defined by the normalization of the measure $T_*\mu$:

$$T_+\mu = \frac{T_*\mu}{||T_*\mu||} = \frac{T_*\mu}{\mu(M_{-1})} \tag{2.4}$$

Definition. A measure μ on M is said to be conditionally invariant under T if $T_+\mu = \mu$. Obviously, any conditionally invariant measure μ is supported on M_+, and there is a $\lambda > 0$ such that $\mu(T^{-1}A \cap M_+) = \lambda\mu(A \cap M_+)$ for any $A \subset M$.

Theorem 1. *Assume the mixing condition. The map T has a unique conditionally invariant probability measure μ_+ whose conditional measures on unstable fibers are Hölder continuous. In fact, those conditional measures are u-SBR measures ν_U^u, $U \in \mathcal{U}_+$. For any smooth measure μ on M (see again the above convention) the sequence $T_+^n\mu$ weakly converges, as $n \to \infty$, to the measure μ_+. Furthermore, the sequence $\lambda_+^{-n} \cdot T_*^n\mu$ weakly converges, as $n \to \infty$, to the measure $c[\mu] \cdot \mu_+$, where $c[\mu] > 0$ is a linear functional on smooth measures on M.*

Remark. The conditionally invariant measure μ_+ constructed in this way is physically natural according to the original Pianigiani-Yorke motivation [**10**]. This measure coincides with the Sinai-Bowen-Ruelle measure in the case $H = \varnothing$.

Corollary 1. *Let $U \in \mathcal{U}$. If μ is a singular measure supported on U with Hölder continuous density (on U), then the sequence $T_+^n \mu$ weakly converges to μ_+.*

We call $\gamma_+ = \ln \lambda_+^{-1}$ the *escape rate*, cf. [**6,8,7,4**].

Next, since the set M_+ is invariant under T^{-1}, the measures $T_*^{-n}\mu_+$ for $n \geq 1$ are probability measures for all $n \geq 0$. In virtue of Theorem 1 they coincide with the conditional measures $\mu_+(\cdot/M_{-n})$ satisfying

$$\mu_+(A/M_{-n}) = \mu_+(A \cap M_{-n})/\mu_+(M_{-n}) = \lambda_+^{-n} \cdot \mu_+(A \cap M_{-n}) \qquad (2.5)$$

Theorem 2. *The sequence of measures $T_*^{-n}\mu_+ = \mu_+(\cdot/M_{-n})$ weakly converges, as $n \to \infty$, to a probability measure, η_+, supported on the set $\Omega = M_+ \cap M_-$. The measure η_+ is T-invariant, i.e.*

$$\eta_+(T^{-1}A) = \eta_+(TA) = \eta_+(A) \qquad (2.6)$$

for every Borel set $A \subset M$.

Theorem 3. *The measure η_+ is an equilibrium state for the Hölder continuous potential*

$$g_+(x) = -\log J^u(x) \qquad (2.7)$$

on Ω and its topological pressure is $P(\eta_+) = -\log \lambda_+^{-1} = -\gamma_+$. Thus, η_+ is a Gibbs measure. The sum of positive Lyapunov exponents of the map T is

$$\chi_{\eta_+}^+ = \int_\Omega \log J^u(x)\, d\eta_+(x) \qquad a.e. \qquad (2.8)$$

The variational principle

$$\begin{aligned} -\gamma_+ &= h_{\eta_+}(T) - \int_\Omega \log J^u(x)\, d\eta_+(x) \\ &= \sup_\eta \{h_\eta(T) - \int_\Omega \log J^u(x)\, d\eta(x)\} \end{aligned} \qquad (2.9)$$

holds, where $h_{\eta+}(T)$ denotes the Kolmogorov-Sinai entropy of the measure $\eta+$, and the supremum is taken over all T-invariant probability *measures on* Ω. *The left equation in (2.9) is equivalent to the following escape rate formula*

$$\chi^+_{\eta_+} = h_{\eta_+}(T) + \gamma_+ \tag{2.10}$$

3. Symbolic dynamics and new results for the mixing case

We translate Theorems 1-3 into the language of symbolic dynamics to obtain new properties of the measures μ_+ and η_+ under the mixing condition.

Define a transition matrix $A' = (A'_{ij})$ of size $I' \times I'$ by

$$A'_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} R_i \cap T^{-1}(int\, R_j) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}.$$

In the space $\Sigma' = \{1, 2, \dots, I'\}^{\mathbb{Z}}$ of doubly infinite sequences $\underline{\omega} = \{\omega_i\}_{-\infty}^{\infty}$ with the product topology we consider a closed subset

$$\Sigma'_{A'} = \{\underline{\omega} \in \Sigma' : A'_{\omega_i \omega_{i+1}} = 1 \;\; for\ all \;\; -\infty < i < \infty\}$$

The left shift homeomorphism $\sigma : \Sigma'_{A'} \to \Sigma'_{A'}$ is defined by $(\sigma(\underline{\omega}))_i = \omega_{i+1}$. This symbolic system is a subshift of finite type.

There is a natural projection $\Pi : \Sigma'_{A'} \to M'$, continuous, surjective and commuting with the dynamics: $\Pi \circ \sigma = T \circ \Pi$. This projection is one-to-one on the set $M' \setminus \cup_{j \in \mathbb{Z}} T^j(\partial R')$.

The partition $\mathcal{R} = \{R_1, \dots, R_I\}$ of $M = M' \setminus H$ defines a $I \times I$ submatrix $A = (A_{ij})$ of A'. We call A the transition matrix for the restriction of T on M. It defines a new subshift of finite type by

$$\Sigma_A = \{\underline{\omega} \in \Sigma'_{A'} : \omega_i \leq I \;\; for\ all \;\; -\infty < i < \infty\}.$$

It is clear that $\Pi(\Sigma_A) = \Omega$.

Consider also a 'hybrid' symbolic space

$$\Sigma_+ = \{\underline{\omega} \in \Sigma'_{A'} : \omega_i \leq I \;\; for\ all \;\; i \leq 0\}.$$

Its positive semi-sequences $\{\omega_i\}_{i=1}^{\infty}$ are defined just like those in $\Sigma'_{A'}$, while its negative semi-sequences $\{\omega_i\}_{i=-\infty}^{0}$ are defined in the same way as those in Σ_A. This space is not σ-invariant, but it is σ^{-1}-invariant. It is easy to check that $\Pi(\Sigma_+) = M_+$.

Fact. [**13,1,11**]. Every topologically transitive Anosov diffeomorphism $T : M' \to M'$ of class $C^{1+\alpha}$ has a unique SBR measure μ^u. Its conditional measures on unstable fibers are absolutely continuous with Hölder continuous densities. The measure μ^u is a weak limit of the iterates of any smooth measure on M' under T^n as $n \to \infty$. For any Markov partition $\mathcal{R}'$ with sufficiently small rectangles the measure $\bar{\mu}^u = \Pi^{-1} \circ \mu^u$ on the symbolic space $\Sigma'_{A'}$ is a Gibbs measure with potential function $\bar{g}(\underline{\omega}) = -\log J^u(\Pi(\underline{\omega}))$ and topological pressure $P = 0$.

The measure μ^u conditioned on M is smooth under our convention in Section 2. Thus, $T_+^n \mu^u$ weakly converges, as $n \to \infty$, to μ_+. Since μ^u is T-invariant on M', we actually have $T_+^n \mu^u = \mu^u(\cdot/M_n)$. Therefore, the measure μ^u conditioned on M_n approaches, as $n \to \infty$, the measure μ_+ on M_+. It follows from the results in [4] (see Corollary 5.7 there) that there are constants $C_1, C_2 > 0$ such that for all $n \geq 0$

$$C_1 \leq \mu^u(M_n)/e^{n\gamma_+} \leq C_2$$

Combining the above facts gives the following property of the measure $\bar{\mu}_+ = \Pi^{-1}\mu_+$ on the symbolic space Σ_+.

Theorem 4. *For any admissible cylinder $C = (\omega_{-n}, \dots, \omega_m) \subset \Sigma_+$ and every symbolic sequence $\underline{\omega} \in C$ we have*

$$C_3 \leq \frac{\bar{\mu}_+(C)}{\exp\left(\sum_{i=-n}^{m} \bar{g}(\sigma^i \underline{\omega}) + n\gamma_+\right)} \leq C_4 \tag{3.1}$$

where $C_3, C_4 > 0$ are constants independent of the cylinder C or the values of n, m.

Comparing this theorem to Bowen's definition of Gibbs measures [1] suggests us to call the measure $\bar{\mu}_+$ on Σ_+ a 'hybrid' Gibbs measure with the potential function $\bar{g}(\underline{\omega})$. Unlike Bowen's definition, however, here the 'positive' and 'negative' components of the cylinder C have different 'topological pressures', $P_+ = 0$ and $P_- = -\gamma_+$ respectively.

For any $n \geq 1$ the measure $\sigma_*^{-n}\bar{\mu}_+$ is supported on $\sigma^{-n}\Sigma_+$. This space has the same cylinders of length $2n+1$, i.e. $(\omega_{-n}, \dots, \omega_n)$, as the space Σ_A. It is clear that $\sigma_*^{-n}\bar{\mu}_+$ converges, as $n \to \infty$, to the Gibbs measure $\bar{\eta}_+ = \Pi^{-1}\eta_+$ on Σ_+ corresponding to the same potential function $\bar{g}(\underline{\omega})$. This is exactly what Theorems 2 and 3 say.

Lastly, let $C = (\omega_0, \dots, \omega_k)$ be any admissible cylinder of length $k+1$ in $\Sigma'_{A'}$, and let $\omega_0 \le I$. Denote by $\bar{\mu}_{+,C}$ the measure $\bar{\mu}_+$ conditioned on $\Sigma_+ \cap C$. Its inverse images $\sigma_*^{-n}(\bar{\mu}_{+,C})$ behave asymptotically, as $n \to \infty$, just like $\sigma_*^{-n}(\bar{\mu}_+)$, because the cylinder C is moved under σ^{-n} to the right and eventually its influence vanishes. Thus, the measure $\sigma_*^{-n}(\bar{\mu}_{+,C})$ weakly converges to the same Gibbs measure $\bar{\eta}_+$. Back on M, this last conclusion means the following.

Corollary 2. *Let R' be any s-inscribed subrectangle in any rectangle $R_i \in \mathcal{R}$ (i.e., R' is a union of some stable fibers $S \in \mathcal{S}, S \subset R_i$). Denote by $\mu_{+,R'}$ the measure μ_+ conditioned on $R' \cap M_+$. Then the sequence $T_*^{-n}\mu_{+,R'}$ weakly converges, as $n \to \infty$, to the measure η_+.*

This corollary is dual to Corollary 1, for it shows that the measure η_+ can be obtained by backward iterations of a measure supported on just one stable fiber, $S \cap M_+$, the latter measure is μ_+ conditioned on $S \cap M_+$. This corollary was missing in Ref. [**4**], and we need it in this paper.

4. Topologically transitive case

Here we replace the mixing condition by the following weaker one.

Transitivity condition. The symbolic dynamics generated by the partition $\mathcal{R} = \{R_1, \dots, R_I\}$ of M is a topologically transitive subshift of finite type. Equivalently, for any $R_i, R_j \in \mathcal{R}$ there is a $k_{ij} \ge 1$ such that $\mathrm{int} R_i \cap T^{k_{ij}}(R_j \cap M_{-k_{ij}}) \ne \varnothing$.

Under this condition the subshift is either topologically mixing (i.e. T satisfies the mixing condition) or periodic. The latter means that there is a finite $p \ge 2$ (period) and a partition of $\mathcal{R}$ into p subgroups $\mathcal{R}_1, \dots, \mathcal{R}_p$ cyclically permuted by the shift. Precisely, $\mathrm{int} R_i \cap T(R_j \cap M_{-1}) \ne \varnothing$ if and only if $R_i \in \mathcal{R}_l$ and $R_j \in \mathcal{R}_{l+1}$ for some l (here and on l is a cyclic index, i.e. $l = p+1$ is identified with $l = 1$). Besides, the map T^p restricted to $M^{(l)} = \cup_{R \in \mathcal{R}_l} R$ for any l satisfies the mixing assumption.

The map T^p restricted to $M^{(l)}$ has all the properties listed in the previous section. In particular, there are conditionally invariant measures $\mu_+^{(l)}$ on $M_+^{(l)} = M_+ \cap M^{(l)}$ and T^p-invariant measures $\eta_+^{(l)}$ on the sets $\Omega^{(l)} = \Omega \cap M^{(l)}$. We call these *basic measures.* These measures satisfy Theorems 1-3 with T replaced by T^p and M by $M^{(l)}$.

It is standard in the ergodic theory to reduce transitive but nonmixing subshifts to mixing ones by replacing T with its appropriate iterate, T^p. It is interesting, however, to extend Theorems 1-3 directly to the nonmixing map T, the task we accomplish in this section.

According to Theorem 1, every basic measure $\mu_+^{(l)}$ is a weak limit of $c[\mu] \cdot \left[\lambda_+^{(l)}\right]^{-n} T_*^{pn}\mu$, as $n \to \infty$, for any smooth measure μ on $M^{(l)}$. It is then clear that the eigenvalues of the measures $\mu_+^{(l)}$ under T^p coincide, i.e. $\lambda_+^{(l)} = \bar{\lambda}_+$ for all l. Also, for any l the measure $T_*\mu_+^{(l)}$ is proportional to $\mu_+^{(l+1)}$, i.e. $T_*\mu_+^{(l)} = \lambda_l \mu_+^{(l+1)}$ with some $\lambda_l \in (0,1]$. Then $\bar{\lambda}_+ = \lambda_1 \cdots \lambda_p$. From these remarks and the cyclic character of the map T we derive the following.

Theorem 5. *There is a unique conditionally invariant measure μ_+ for the map T, whose conditional measures on unstable fibers are smooth. These are, in fact, the u-SBR measures ν_U^u. The eigenvalue of μ_+ is $\lambda_+ = (\bar{\lambda}_+)^{1/p}$. The measure μ_+ is a weighted sum of basic measures*

$$\mu_+ = w_1\mu_+^{(1)} + \cdots + w_p\mu_+^{(p)}$$

where the weights $w_l > 0$ are uniquely determined by the equations $w_l\lambda_l = w_{l+1}\lambda_+$ for all l and $w_1 + \cdots + w_p = 1$.

Example. Let $p = 2$, and $\lambda_1 = 1$, $\lambda_2 = 1/4$. Then the eigenvalue of the measure μ_+ is $1/2$ and the weights are $w_1 = 1/3$ and $w_2 = 2/3$.

However, the images $T_+^n\mu$ of an arbitrary smooth measure μ on M generally need not converge, as $n \to \infty$, to the measure μ_+. Normally, the sequence $T_+^n\mu$ periodically approaches a finite number ($\leq p$) of limit measures, all of them being some weighted sums of the basic measures $\mu_+^{(1)}, \ldots, \mu_+^{(p)}$ (in particular, they are all equivalent to μ_+). The Cesaro limit of the sequence $T_+^n\mu$ always exists and does not depend on μ. But it is an equidistributed sum of the basic measures

$$\mu_+^0 = \frac{1}{p}\left(\mu_+^{(1)} + \cdots + \mu_+^{(p)}\right)$$

Even though this measure is equivalent to μ_+, it is generally different from μ_+.

Theorem 6. (i) *The equidistributed sum of basic measures* $\eta_+^{(p)}$,

$$\eta_+ = \frac{1}{p}\left(\eta_+^{(1)} + \cdots + \eta_+^{(p)}\right)$$

is a T*-invariant measure on* Ω. *It is the only* T*-invariant measure equivalent to* $\eta_+^{(l)}$ *on* $M^{(l)}$ *for every* l.
(ii) *The weak Cesaro limit of the sequence* $T_*^{-n}\mu_+$, *as* $n \to \infty$, *is* η_+.

Proof. The basic measures $\eta_+^{(l)}$ on $\Omega^{(l)}$ satisfy the invariance property $T_*\eta_+^{(l)} = \eta_+^{(l+1)}$. This follows from Theorem 2 and Corollary 2. Then the part (i) is immediate.

It is easy to see that the measure η_+ is the weak limit of the sequence $T_*^{-n}\mu_+^0$ as $n \to \infty$. However, the sequence $T_*^{-n}\mu_+ = \mu_+(\cdot/M_{-n})$ generally does not converge to any measure on Ω. Instead, it periodically approaches a finite number ($\leq p$) of measures on Ω that will be weighted sums of the basic measures $\eta_+^{(l)}$. In the above example, the two limit measures for the sequence $T_*^{-n}\mu_+$ have weight distributions $(1/3, 2/3)$ and $(2/3, 1/3)$. It is now clear that the part (ii) of Theorem 6 holds.

Theorem 7. (i) *The measure* η_+ *is a Gibbs measure with potential function* $g(x) = -\ln J^u(x)$ *and topological pressure* $P = \ln\lambda_+$.
(ii) *It satisfies the equation (2.10).*

Proof. According to Theorem 3 the basic measures $\eta_+^{(l)}$ are Gibbs with potential $g_l(x) = -\ln(J^u(x)\cdots J^u(T^{p-1}x))$, $x \in \Omega^{(l)}$, and topological pressure $P_l = \ln\bar{\lambda}_+$. Then the part (i) easily follows.

The equation (2.10) holds for the measure $\eta_+^{(l)}$ and the map T^p on $\Omega^{(l)}$. It is easy to check that every quantity involved in (2.10) decreases by a factor of p if we replace $T^p : \Omega^{(l)} \to \Omega^{(l)}$ by $T : \Omega \to \Omega$ and $\eta_+^{(l)}$ by η_+. This gives the part (ii).

Summarizing, we find that Theorems 1- 3 still hold under the transitivity assumption, with two exceptions. First, the images $T_+^n\mu$ of an arbitrary smooth measure μ on M do not exactly converge to μ_+. They approach a finite number of limit measures on M_+, all equivalent to μ_+. The same goes to the sequence $T_*^{-n}\mu_+$ and the limit measure η_+. In the latter case, however, the Cesaro limit of $T_*^{-n}\mu_+$ is always η_+. Corollaries 1 and 2 cannot be extended to nonmixing cases.

5. Nonrecurrent rectangles

One can classify the rectangles $R \in \mathcal{R}$ just like states of Markov chains are classified in probability theory. We call a rectangle $R \in \mathcal{R}$ recurrent if its interior points come back to R under T, i.e. $\text{int}R \cap T^n(R \cap M_{-n}) \neq \varnothing$ for some $n \geq 1$. In the trivial case, where all the rectangles are nonrecurrent (transient), the sets M_+, M_- and Ω are all empty, and the phase space M 'escapes' entirely.

The recurrent rectangles can be grouped, within each group points from any rectangle are eventually mapped into any other rectangle. The symbolic dynamics within every group is a topologically transitive (TT) subshift of finite type.

In this section we still assume that there is just one transitive group of recurrent rectangles $R_1, \cdots, R_{I_0}$, but we allow some nonrecurrent rectangles $R_{I_0+1}, \ldots, R_I$ as well. Put $M^{(1)} = R_1 \cup \ldots \cup R_{I_0}$.

Nonrecurrent rectangles are further subdivided into three groups:
(i) incoming: such that $\text{int}T^n(R_i \cap M_{-n}) \cap M^{(1)} \neq \varnothing$ for some $n \geq 1$;
(ii) outgoing: such that $\text{int}T^{-n}(R_i \cap M_n) \cap M^{(1)} \neq \varnothing$ for some $n \geq 1$;
(iii) isolated: such that $\text{int}T^n(R_i \cap M_{-n}) \cap M^{(1)} = \varnothing$ for all $n \in \mathbb{Z}$.

The set of nonwandering points Ω obviously belongs in $M^{(1)}$. The restriction of the map T to $M^{(1)}$ satisfies the transitivity assumption in the previous section. Thus, there is a conditionally invariant measure $\mu_+^{(1)}$ on $M_+^{(1)} = M_+ \cap M^{(1)}$ with eigenvalue $\lambda_+^{(1)}$, and the corresponding T-invariant measure $\eta_+^{(1)}$ on Ω.

The isolated rectangles escape to H altogether in a finite time and have no influence on the measures μ_+, η_+ whatsoever. The incoming rectangles are absorbed into $M^{(1)}$ in a finite time, so their presence (or absence) cannot affect the properties of the measures μ_+ or η_+ either.

The set M_+ intersects only recurrent and outgoing rectangles. The measures $T_*^n\mu_+^{(1)}$ (the images of $\mu_+^{(1)}$ under the maps T^n on M) will be supported on M_+, and their restrictions to $M_+^{(1)}$ will be always proportional to $\mu_+^{(1)}$. It is then easy to check the following.

Theorem 8. *Under the above conditions, there is a unique conditionally invariant measure μ_+ for the map T supported on M_+ with absolutely continuous conditional measures on unstable fibers. They are, in fact, the u-SBR measures ν_U^u. The measure μ_+ is proportional to $\mu_+^{(1)}$ on the set $M_+^{(1)}$. These two measures have the same eigenvalue $\lambda_+ = \lambda_+^{(1)}$.*

If the transitive group of rectangles is topologically mixing, the sequence $T_+^n\mu$ converges, as $n \to \infty$, to μ_+ for any smooth measure μ on M. In nonmixing cases the situation is equivalent to the one in the previous section.

Theorem 9. (i) *The T-invariant measure η_+ on Ω simply coincides with the measure $\eta_+^{(1)}$. Thus, it enjoys all the properties established by Theorems 3 and 3.*
(ii) *The measure η_+ is the weak Cesaro limit of $T_*^{-n}\mu_+$ as $n \to \infty$ (in the mixing case, it is just the weak limit).*

Proof. Only the part (ii) needs a proof. According to Theorem 6, the weak Cesaro limit of the sequence $T_*^{-n}\mu_+^{(1)}$ is η_+. Consider the measure μ_+ conditioned on outgoing rectangles. It will be transferred under T_*^{-n} into measures supported on some s-inscribed subrectangles in some rectangles $R_i \subset M^{(1)}$ and on those subrectangles those measures will be proportional to $\mu_+^{(1)}$. Due to Corollary 2, such measures converge to η_+, as $n \to \infty$, in the same way as the sequence $T^{-n}\mu_+^{(1)}$. The theorem is proved.

6. Two TT groups of rectangles

As it was remarked in the previous section, recurrent rectangles can be divided into topologically transitive (TT) groups so that in each group points from any rectangle can be mapped into any other rectangle. In this Section we will assume that there are just two TT groups of rectangles: $M^{(1)} = R_1 \cup \cdots \cup R_{I_1}$, $M^{(2)} = R_{I_1+1} \cup \cdots \cup R_{I_2}$, and some non-recurrent rectangles $R_{I_2+1}, \ldots, R_I$. If there is no connection between these groups, i.e. $\mathrm{int}(T^n M^{(1)} \cap T^m M^{(2)}) = \varnothing$ for all $m, n \in \mathbb{Z}$, then we have two trivially independent repellers, one in $M^{(1)}$ and another in $M^{(2)}$.

There may be, however, a one-way route from $M^{(1)}$ to $M^{(2)}$, i.e. $\mathrm{int}(T^n M^{(1)} \cap M^{(2)}) \neq \varnothing$ for some $n \geq 1$. In this case the picture gets more intricate. The rate of escape from $M^{(1)}$ is still the same as for the map $T|_{M^{(1)}}$, as if $M^{(2)}$ did not exist. The escape from $M^{(2)}$, however, is combined with the influx from $M^{(1)}$. The resulting escape rate from M and conditionally invariant measures will be then determined by three factors: the escape of mass from $M^{(1)}$, $M^{(2)}$ and the transfer of mass from $M^{(1)}$ to $M^{(2)}$.

Nonrecurrent rectangles are now subdivided into four groups:

(i) incoming: such that $\mathrm{int} T^n(R_i \cap M_{-n}) \cap (M^{(1)} \cup M^{(2)}) \neq \varnothing$ for some $n \geq 1$ but $\mathrm{int} T^{-n}(R_i \cap M_n) \cap (M^{(1)} \cup M^{(2)}) = \varnothing$ for all $n \geq 1$;

(ii) outgoing: such that $\mathrm{int} T^{-n}(R_i \cap M_n) \cap (M^{(1)} \cup M^{(2)}) \neq \varnothing$ for some $n \geq 1$ but $\mathrm{int} T^n(R_i \cap M_{-n}) \cap (M^{(1)} \cup M^{(2)}) = \varnothing$ for all $n \geq 1$;

(iii) isolated: such that $\mathrm{int} T^n(R_i \cap M_{-n}) \cap (M^{(1)} \cup M^{(2)}) = \varnothing$ for all $n \in \mathbb{Z}$;

(iv) transmitting: such that $\mathrm{int}\, T^n(R_i \cap M_{-n}) \cap M^{(2)} \neq \varnothing$ for some $n \geq 1$ and $\mathrm{int}\, T^{-m}(R_i \cap M_m) \cap M^{(1)} \neq \varnothing$ for some $m \geq 1$.

Slightly abusing the language, we will say that incoming rectangles have one-way connections *to* $M^{(1)} \cup M^{(2)}$, and the outgoing rectangles have one-way connections *from* $M^{(1)} \cup M^{(2)}$. We may also say that transmitting rectangles are connected *from* $M^{(1)}$ and *to* $M^{(2)}$.

For the map T restricted to $M^{(i)}$, $i = 1, 2$, we denote by $M_\pm^{(i)}$ and $\Omega^{(i)}$ the corresponding sets defined as in Introduction, and by $\mu_\pm^{(i)}$ and $\eta_\pm^{(i)}$ their conditionally invariant and invariant measures, respectively.

We denote by $\lambda_{\pm}^{(i)}$, $i = 1, 2$, the corresponding eigenvalues.

It is clear that

$$M_+ = \cup_{n\geq 0} T^n (M_+^{(1)} \cap M_+^{(2)})$$

and

$$M_- = \cup_{n\geq 0} T^{-n} (M_-^{(1)} \cap M_-^{(2)}).$$

In the present case, the set M_+ consists, in addition to $M_+^{(1)} \cup M_+^{(2)}$, of some unstable fibers in outgoing and transmitting rectangles, as well as some unstable fibers in $M^{(2)}$ not included in $M_+^{(2)}$. These fibers are images of $M_+^{(1)}$ under T^n, $n \geq 1$, and they are getting closer and closer to $M_+^{(2)}$ as $n \to \infty$. Symmetric statements can be made of the set M_-.

The set of nonwandering points $\Omega = M_+ \cap M_-$ includes $\Omega^{(1)}$ and $\Omega^{(2)}$, but is not limited to them. It also contains (i) points of intersection of stable fibers of $M_-^{(2)}$ and unstable fibers of $(T^n M_+^{(1)}) \cap M^{(2)}$, as well as (ii) similar points in $M^{(1)}$, and (iii) the points in transmitting rectangles which belong in $T^n M_+^{(1)} \cap T^{-m} M_-^{(2)}$, $n, m \geq 1$.

Standing assumption for all theorems in Sections 6 and 7. The map T restricted to every TT component is not only topologically transitive, but also topologically mixing.

This is assumed for simplicity only. Nonmixing maps require the modifications to our results completely described in Section 4.

We now construct the conditionally invariant measure μ_+ for T on M. Obviously, isolated and incoming rectangles do not affect the measure μ_+. Outgoing and transmitting rectangles can capture some fraction of this measure as we described in the previous section. A new twist here is a flow of mass into $M^{(2)}$ from the transmitting rectangles or directly from $M^{(1)}$. The flowing mass then evolves in $M^{(2)}$ and approaches $M_+^{(2)}$ competing with the measure $\mu_+^{(2)}$. This flow is characterized by parameters described below.

Denote by $M^{(1+)}$ the union of $M^{(1)}$ with all outgoing rectangles connected *from* $M^{(1)}$ and all transmitting rectangles. Consider the restriction of the map T to $M^{(1+)}$. This restriction has one TT group of rectangles (it is $M^{(1)}$) and others act just like outgoing rectangles in the previous section. We proved there that T on $M^{(1+)}$ has a conditionally invariant measure $\mu_+^{(1+)}$ with the same eigenvalue $\lambda_+^{(1)}$ as the measure $\mu_+^{(1)}$. Similarly, denote by $M^{(2+)}$ the union of $M^{(2)}$ with all outgoing rectangles connected *from* $M^{(2)}$. Let $\mu_+^{(2+)}$ be the conditional invariant measure for the restriction of T to $M^{(2+)}$. Note that, in a peculiar way, the sets $M^{(1+)}$ and $M^{(2+)}$ may have common outgoing rectangles. But even in this case the measures $\mu_+^{(1+)}$ and $\mu_+^{(2+)}$ are supported on disjoint closed sets.

Now, let $q_1^{(12)} > 0$ be the fraction of $\mu_+^{(1+)}$ transmitted to $M^{(2+)}$ under the action of T, i.e. $q_1^{(12)} = T_*\mu_+^{(1+)}(M^{(2+)})$. Denote by $\mu_1^{(12)}$ the measure $T_*\mu_+^{(1+)}$ conditioned on $M^{(2+)}$. For any $k \geq 2$ let $q_k^{(12)} > 0$ be the fraction of $\mu_+^{(1+)}$ transmitted to $M^{(2+)}$ and surviving $k-1$ iterations of T within $M^{(2+)}$, i.e. $q_k^{(12)} = q_1^{(12)} T_*^{k-1}\mu_1^{(12)}(M^{(2+)})$. For any $k \geq 2$ let $\mu_k^{(12)} = T_+^{k-1}\mu_1^{(12)}$. The measure $\mu_1^{(12)}$ is supported on some unstable fibers in $M^{(2+)}$. Its further evolution under T_+^k, $k \geq 1$, within $M^{(2+)}$ will satisfy Theorem 8. According to that theorem, $\mu_k^{(12)}$ will weakly converge to $\mu_+^{(2+)}$ as $k \to \infty$, and $q_k^{(12)} \sim [\lambda_+^{(2)}]^k$, i.e. $q_k^{(12)}[\lambda_+^{(2)}]^{-k} \to$ *const* > 0 as $k \to \infty$.

Theorem 10. *Assume that the two TT groups of rectangles are topologically mixing.*

(i) *If* $\lambda_+^{(1)} > \lambda_+^{(2)}$, *then there are two conditionally invariant probability measures for* T *whose conditional measures on unstable fibers are u-SBR measures. One coincides with* $\mu_+^{(2+)}$ *and has eigenvalue* $\lambda_+^{(2)}$. *The other has eigenvalue* $\lambda_+^{(1)}$, *it is a weighted sum*

$$\mu_+ = Q^{-1} \cdot \left(\mu_+^{(1+)} + \sum_{k=1}^{\infty} q_k^{(12)} [\lambda_+^{(1)}]^{-k} \mu_k^{(12)} \right) \tag{6.1}$$

where Q^{-1} *is the normalization factor:*

$$Q = 1 + \sum_{k=1}^{\infty} q_k^{(12)} [\lambda_+^{(1)}]^{-k}$$

In particular, $\mu_+(M^{(2+)}) = 1 - Q^{-1}$ *and* $\mu_+(M_+^{(2)}) = 0$.

(ii) *If* $\lambda_+^{(1)} \leq \lambda_+^{(2)}$, *then the only conditionally invariant probability measure for* T *with u-SBR conditional distributions on unstable fibers is* $\mu_+^{(2+)}$ *with eigenvalue* $\lambda_+^{(2)}$.

For any smooth measure μ *on* M *the sequence* $T_+^n \mu$ *weakly converges, as* $n \to \infty$, *to one of the above conditionally invariant measures. In the case (i) this limit measure is the one from (6.1) (rather than* $\mu_+^{(2+)}$*) if and only if* $\mu(M^{(1-)}) \neq 0$, *where* $M^{(1-)}$ *is the union of* $M^{(1)}$ *and all incoming rectangles connected to* $M^{(1)}$.

Proof. It is enough to investigate the evolution under T_* of the measure $\mu_0 = x\mu_+^{(1+)} + y\mu_+^{(2+)}$ with arbitrary $x, y \geq 0$, $x + y = 1$. Its image, $T_*\mu_0$, is

$$x\lambda_+^{(1)}\mu_+^{(1+)} + xq_1^{(12)}\mu_1^{(12)} + y\lambda_+^{(2)}\mu_+^{(2+)}$$

Its k-th image, $T_*^k\mu_0$, is

$$x[\lambda_+^{(1)}]^k\mu_+^{(1+)} + x\sum_{i=1}^{k} q_i^{(12)}[\lambda_+^{(1)}]^{k-i}\mu_i^{(12)} + y[\lambda_+^{(2)}]^k\mu_+^{(2+)} \tag{6.2}$$

The norm of the second term in (6.2) is

$$x[\lambda_+^{(1)}]^k \sum_{i=1}^{k} c_i[\lambda_+^{(2)}/\lambda_+^{(1)}]^i$$

with some $c_i = O(1)$, i.e. c_i are bounded away from 0 and ∞. This series converges iff $\lambda_+^{(1)} > \lambda_+^{(2)}$. In this case the asymptotics of $T_*^k\mu_0$ will be determined by the first two terms in (6.2) provided $x \neq 0$ and by the third term alone otherwise. In the case $\lambda_+^{(1)} \leq \lambda_+^{(2)}$ we use the convergence of $\mu_k^{(12)}$ to $\mu_+^{(2+)}$. Renormalizing and taking limit as $k \to \infty$ prove the theorem.

Theorem 11. (i) *If* $\lambda_+^{(1)} > \lambda_+^{(2)}$, *then the measure* $\eta_+ = \lim T_*^{-n}\mu_+$ *is either* $\eta_+^{(1)}$ *or* $\eta_+^{(2)}$ *depending on* μ_+ *being defined by (6.1) or being equal to* $\mu_+^{(2+)}$;

(ii) *If* $\lambda_+^{(1)} \leq \lambda_+^{(2)}$, *then* $\eta_+ = \eta_+^{(2)}$.

In either case η_+ *is a* T*-invariant Gibbs measure with potential function* $g(x) = -\ln J^u(x)$ *and topological pressure* $P = \ln\lambda_+$. *It satisfies the* *equation (2.10).*

This theorem readily follows from the previous one, in view of Theorems 2 and 3.

In the case (i) of Theorems 10 and 11, the options $\mu_+ = \mu_+^{(2+)}$ and $\eta_+ = \eta_+^{(2)}$ can be regarded as quite singular. Indeed, these measures are generated by initially smooth measures μ such that $\mu(M^{(1-)}) = 0$. ¿From now on, we will rule out such degenerate measures:

Definition. The measures μ_+ and η_+ are said to be regular if they are generated by smooth measures on M that are positive on every open set.

In particular, $\mu(M^{(1-)}) > 0$, so that in each case in Theorems 10 and 11 the regular measures are unique. We will restrict ourselves to regular measures. Then these theorems can be summarized as follows.

Rule 1. The eigenvalue λ_+ of the map T on M equals the largest of the eigenvalues of T restricted to TT components. The conditionally invariant measure μ_+ is determined by that of the component with the largest eigenvalue. The other components that have one-way connections *to* the one with the largest eigenvalue, play no role. The other components that have one-way connections *from* the one with the largest eigenvalue, play the same role as outgoing rectangles, capturing a fraction of μ_+. The invariant measure η_+ coincides with the one on the TT component with the largest eigenvalue, as if the others did not exist.

Motivated by this rule, we will call the TT components with the largest eigenvalue (i.e., the smallest escape rate) the *dominating* components. We will see later that the above rule holds for maps with any number of TT components, provided the dominating component is unique. Necessary corrections in the case of several dominating components will be made below.

7. Three TT groups of rectangles

The description of measures μ_+ and η_+ gets more complicated in the case of more than two TT groups of rectangles. However, the entire picture is still determined by the rates of escape of mass from every transitive component and by the rates of transfer of mass between components.

It is clear that there can be only one-way routes between components. These routes make an oriented graph in which transitive components are vertices. Moreover, there can be no (oriented) loops in this graph, so that it is actually a tree. We can assume that it is a connected tree, otherwise it decomposes into two or more trivially independent trees.

In this section we study maps with three transitive components. Again, for simplicity we assume that all these components are topologically mixing. Let us denote them by $M^{(1)} = R_1 \cup \ldots \cup R_{I_1}$, $M^{(2)} = R_{I_1+1} \cup \ldots \cup R_{I_2}$ and $M^{(3)} = R_{I_2+1} \cup \ldots \cup R_{I_3}$. In addition, there may be some non-recurrent rectangles $R_{I_3+1}, \ldots, R_I$. For the map T restricted to $M^{(i)}$, $i = 1, 2, 3$, we will use the notations $M_+^{(i)}$, $\lambda_\pm^{(i)}$, $\mu_\pm^{(i)}$ and $\eta_\pm^{(i)}$ introduced in the previous section.

There are four nonisomorphic connected oriented trees with three vertices. They are (up to renumbering of vertices):
(I) $M^{(1)} \to M^{(2)} \to M^{(3)}$,
(II) $M^{(1)} \to M^{(2)}$ and $M^{(1)} \to M^{(3)}$,
(III) $M^{(1)} \to M^{(2)} \to M^{(3)}$ and $M^{(1)} \to M^{(3)}$,
(IV) $M^{(1)} \to M^{(3)}$ and $M^{(2)} \to M^{(3)}$.

For every of these configurations, the nonrecurrent rectangles can be classified and the sets $M_\pm$ and Ω can be described in a way similar to the one we gave in the previous section. We do not dwell on this, since it will not be essential to our analysis. We turn to the study of the measures μ_+ and η_+ for the map T on M.

The first configuration logically reduces to the study of two TT groups if we consider first the subgroup $M^{(2)} \to M^{(3)}$ independently of $M^{(1)}$ and then the pair $M^{(1)}$ and $M^{(2)} \cup M^{(3)}$. The results will then perfectly fit Rule 1 at the end of the previous section. We leave out the details and turn to the more interesting configurations (II)-(IV).

The configurations (II) and (III) are characterized by flows of mass from $M^{(1)}$ into $M^{(2)}$ and $M^{(3)}$ (directly or via transmitting nonrecurrent rectangles). The flowing mass then evolves in both $M^{(2)}$ and $M^{(3)}$ approaching the sets $M_+^{(2)}$ and $M_+^{(3)}$ respectively. In the configuration (III) there is also a flow of mass from $M^{(2)}$ to $M^{(3)}$. These flows are characterized by parameters described below.

For simplicity, we assume that there are no outgoing or transmitting rectangles in the system. If there are any, one has to take unions of $M^{(i)}$ with outgoing and transmitting rectangles connected *from* $M^{(i)}$ like we did in the previous section. This amounts to somewhat heavier notations but makes little difference in our arguments.

For any pair of components $M^{(i)}$ and $M^{(j)}$ we introduce a sequence of numbers $\{q_k^{(ij)}\}$ similarly to the sequence $\{q_k^{(12)}\}$ in the previous section. Let $q_1^{(ij)} > 0$ be the fraction of $\mu_+^{(i)}$ transmitted to $M^{(j)}$ under the action of T, i.e. $q_1^{(ij)} = T_*\mu_+^{(i)}(M^{(j)})$. Denote by $\mu_1^{(ij)}$ the measure $T_*\mu_+^{(i)}$ conditioned on $M^{(j)}$. For any $k \geq 2$ let $q_k^{(ij)} > 0$ be the fraction of $\mu_+^{(i)}$ transmitted to $M^{(j)}$ and surviving $k-1$ iterations of T restricted to $M^{(j)}$, i.e. $q_k^{(ij)} = q_1^{(ij)} T_*^{k-1}\mu_1^{(ij)}(M^{(j)})$. For any $k \geq 2$ let $\mu_k^{(ij)}$ be the measure $T_*^{k-1}\mu_1^{(ij)}$ conditioned on $M^{(j)}$. The measure $\mu_1^{(ij)}$ is supported on some unstable fibers in $M^{(j)}$. Its further evolution under the restriction of T^k, $k \geq 1$, to $M^{(j)}$ will satisfy Theorem 1. According to that theorem, $\mu_k^{(ij)}$ will weakly converge to $\mu_+^{(j)}$ as $k \to \infty$, and $q_k^{(ij)} \sim [\lambda_+^{(j)}]^k$, i.e. $q_k^{(ij)}[\lambda_+^{(j)}]^{-k} \to const > 0$ as $k \to \infty$.

Theorem 12. *Assume the configuration (II), the mixing condition within every TT group of rectangles, and the absence of outgoing and transmitting rectangles in the system.*

(i) *If* $\lambda_+^{(1)} > \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$, *then the unique regular conditionally invariant measure* μ_+ *has eigenvalue* $\lambda_+^{(1)}$ *and is a weighted sum*

$$\mu_+ = Q^{-1} \cdot \left(\mu_+^{(1)} + \sum_{k=1}^{\infty} q_k^{(12)}[\lambda_+^{(1)}]^{-k}\mu_k^{(12)} + \sum_{k=1}^{\infty} q_k^{(13)}[\lambda_+^{(1)}]^{-k}\mu_k^{(13)} \right) \quad (7.1)$$

where Q^{-1} is the normalization factor:

$$Q = 1 + \sum_{k=1}^{\infty} q_k^{(12)} [\lambda_+^{(1)}]^{-k} + \sum_{k=1}^{\infty} q_k^{(13)} [\lambda_+^{(1)}]^{-k}$$

In particular, $\mu_+(M_+^{(2)} \cup M_+^{(3)}) = 0$.

(ii) *Let* $\lambda_+^{(1)} \leq \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$ *and* $\lambda_+^{(2)} \neq \lambda_+^{(3)}$. *Without loss of generality, assume that* $\lambda_+^{(2)} > \lambda_+^{(3)}$. *Then the only regular conditionally invariant measure* μ_+ *coincides with* $\mu_+^{(2)}$ *and has eigenvalue* $\lambda_+^{(2)}$.

(iii) *If* $\lambda_+^{(1)} \leq \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$ *and* $\lambda_+^{(2)} = \lambda_+^{(3)}$, *then any weighted sum of the measures* $\mu_+^{(2)}$ *and* $\mu_+^{(3)}$ *is a regular conditionally invariant measure for* T. *Its eigenvalue is* $\lambda_+^{(2)} = \lambda_+^{(3)}$.

For any smooth measure μ *on* M *positive on every open set the sequence* $T_+^n \mu$ *weakly converges, as* $n \to \infty$, *to a regular conditionally invariant measure* μ_+. *In the case (iii) the resulting measure* μ_+ *is determined by the initial distribution of* μ *between the TT components.*

Proof. As in the proof of Theorem 10, it is enough to investigate the evolution under T_* of the measure $\mu_0 = x\mu_+^{(1)} + y\mu_+^{(2)} + z\mu_+^{(3)}$ with arbitrary $x, y, z \geq 0$ such that $x + y + z = 1$. Its image, $T_*\mu_0$, is

$$x\lambda_+^{(1)}\mu_+^{(1)} + xq_1^{(12)}\mu_1^{(12)} + y\lambda_+^{(2)}\mu_+^{(2)} + xq_1^{(13)}\mu_1^{(13)} + z\lambda_+^{(3)}\mu_+^{(3)}$$

Its k-th image, $T_*^k\mu_0$, is

$$\begin{aligned} x[\lambda_+^{(1)}]^k\mu_+^{(1)} + x\sum_{i=1}^{k} q_i^{(12)}[\lambda_+^{(1)}]^{k-i}\mu_i^{(12)} + y[\lambda_+^{(2)}]^k\mu_+^{(2)} \\ + x\sum_{i=1}^{k} q_i^{(13)}[\lambda_+^{(1)}]^{k-i}\mu_i^{(13)} + z[\lambda_+^{(3)}]^k\mu_+^{(3)} \end{aligned} \tag{7.2}$$

The rest of the proof goes like that of Theorem 10.

Theorem 13. *Under the conditions of the previous theorem we have*

(i) *If* $\lambda_+^{(1)} > \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$, *then the measure* $\eta_+ = \lim T_*^{-n}\mu_+$ *coincides with* $\eta_+^{(1)}$.

(ii) *Let* $\lambda_+^{(1)} \leq \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$ *and* $\lambda_+^{(2)} > \lambda_+^{(3)}$ *as before. Then* $\eta_+ = \eta_+^{(2)}$.

(iii) *If* $\lambda_+^{(1)} \leq \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$ *and* $\lambda_+^{(2)} = \lambda_+^{(3)}$, *then* η_+ *is a weighted sum of* $\eta_+^{(2)}$ *and* $\eta_+^{(3)}$ *with the same weights as in the case (iii) of the previous theorem.*

In every case η_+ *is a* T*-invariant Gibbs measure with potential function* $g(x) = -\ln J^u(x)$ *and topological pressure* $P = \ln \lambda_+$. *It is ergodic in the cases (i) and (ii), and has two ergodic components in the case (iii). The measure* η_+ *satisfies the equation (2.10).*

We now turn to the configuration (III). A new twist here is a secondary flow of mass from $M^{(1)}$ to $M^{(3)}$ via $M^{(2)}$. For any $m, n \geq 1$ denote by $M_{m,n}^{(1)}$ the set of points of $M^{(1)}$ whose first m images land in $M^{(2)}$ and the following n images land in $M^{(3)}$, i.e.

$$\begin{aligned} M_{m,n}^{(1)} = \{x \in M^{(1)} : T^i x \in M^{(2)} \quad &for\ 1 \leq i \leq m \quad and \\ T^j x \in M^{(3)} \quad &for\ m+1 \leq j \leq m+n\} \end{aligned}$$

Let $r_{m,n}^{(123)} = \mu_+^{(1)}(M_{m,n}^{(1)})$ and $\mu_{m,n}^{(123)} = T_*^{m+n}(\mu_+^{(1)}|M_{m,n}^{(1)})$, note that $\mu_{m,n}^{(123)}$ is a probability measure. Obviously, $T_*^{-1}\mu_{m,1}^{(123)}$ is the measure $\mu_m^{(12)}$ conditioned on $M^{(2)} \cap T^{-1}M^{(3)}$. Therefore, the measure $T_*^{-1}\mu_{m,1}^{(123)}$ converges, as $m \to \infty$, to $\mu_+^{(2)}$ conditioned on $M^{(2)} \cap T^{-1}M^{(3)}$. According to Theorem 1, the measure $\mu_{m,n}^{(123)}$ then converges, as $n \to \infty$, to $\mu_+^{(3)}$, and this convergence is uniform in m. Also, $r_{m,n}^{(123)} \sim [\lambda_+^{(2)}]^m[\lambda_+^{(3)}]^n$, i.e. $r_{m,n}^{(123)}[\lambda_+^{(2)}]^{-m}[\lambda_+^{(3)}]^{-n} \to const > 0$ as $m, n \to \infty$, and the values of $r_{m,n}^{(123)}[\lambda_+^{(2)}]^{-m}[\lambda_+^{(3)}]^{-n}$ are bounded away from 0 and ∞.

Theorem 14. *Assume the configuration (III), the mixing condition within every TT group of rectangles, and the absence of outgoing and transmitting rectangles in the system.*

(i) *If* $\lambda_+^{(1)} > \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$, *then the unique regular conditionally invariant measure* μ_+ *has eigenvalue* $\lambda_+^{(1)}$, *and it is a weighted sum*

$$\begin{aligned} \mu_+ = Q^{-1} \cdot \Big(\mu_+^{(1)} + \sum_{k=1}^{\infty} q_k^{(12)}[\lambda_+^{(1)}]^{-k}\mu_k^{(12)} + \sum_{k=1}^{\infty} q_k^{(13)}[\lambda_+^{(1)}]^{-k}\mu_k^{(13)} \\ + \sum_{m,n=1}^{\infty} r_{m,n}^{(123)}[\lambda_+^{(1)}]^{-m-n}\mu_{m,n}^{(123)}\Big) \end{aligned} \qquad (7.3\ )$$

where Q^{-1} is the normalization factor:

$$Q = 1 + \sum_{k=1}^{\infty} q_k^{(12)} [\lambda_+^{(1)}]^{-k} + \sum_{k=1}^{\infty} q_k^{(13)} [\lambda_+^{(1)}]^{-k} + \sum_{m,n=1}^{\infty} r_{m,n}^{(123)} [\lambda_+^{(1)}]^{-m-n}$$

In particular, $\mu_+(M_+^{(2)} \cup M_+^{(3)}) = 0$.

(ii) *Let $\lambda_+^{(1)} \le \lambda_+^{(2)}$ and $\lambda_+^{(2)} > \lambda_+^{(3)}$. Then the only regular conditionally invariant measure μ_+ has eigenvalue $\lambda_+^{(2)}$ and is a weighted sum*

$$\mu_+ = Q^{-1} \cdot \left(\mu_+^{(2)} + \sum_{k=1}^{\infty} q_k^{(23)} [\lambda_+^{(2)}]^{-k} \mu_k^{(23)} \right) \tag{7.4}$$

where Q^{-1} is the normalization factor:

$$Q = 1 + \sum_{k=1}^{\infty} q_k^{(23)} [\lambda_+^{(2)}]^{-k}$$

In particular, $\mu_+(M_+^{(1)} \cup M_+^{(3)}) = 0$.

(iii) *Let $\lambda_+^{(3)} \ge \max\{\lambda_+^{(1)}, \lambda_+^{(2)}\}$. Then the only regular conditionally invariant measure μ_+ coincides with $\mu_+^{(3)}$ and has eigenvalue $\lambda_+^{(3)}$.*

For any smooth measure μ on M positive on every open set the sequence $T_+^n \mu$ weakly converges, as $n \to \infty$, to a regular conditionally invariant measure μ_+.

Proof. As in the proofs of Theorems 10 and 12, it is enough to investigate the evolution under T_* of the measure $\mu_0 = x\mu_+^{(1)} + y\mu_+^{(2)} + z\mu_+^{(3)}$ with arbitrary $x, y, z \ge 0$ such that $x + y + z = 1$. Its k-th image, $T_*^k \mu_0$, is

$$\begin{aligned} x[\lambda_+^{(1)}]^k \mu_+^{(1)} &+ x \sum_{i=1}^{k} q_i^{(12)} [\lambda_+^{(1)}]^{k-i} \mu_i^{(12)} + y[\lambda_+^{(2)}]^k \mu_+^{(2)} \\ &+ x \sum_{i=1}^{k} q_i^{(13)} [\lambda_+^{(1)}]^{k-i} \mu_i^{(13)} + z[\lambda_+^{(3)}]^k \mu_+^{(3)} \\ &+ y \sum_{i=1}^{k} q_i^{(23)} [\lambda_+^{(2)}]^{k-i} \mu_i^{(23)} + x \sum_{i+j\le k} r_{i,j}^{(123)} [\lambda_+^{(1)}]^{k-i-j} \mu_{i,j}^{(123)} \end{aligned}$$

The rest of the proof goes basically like that of Theorems 10 and 12. In the analysis of the case (ii) the measures $\mu_{i,j}^{(123)}$ play some role. The necessary result follows from two facts: (i) the measure $T_*^{-1}\mu_{i,1}^{(123)}$ converges, as $i \to \infty$, to $\mu_+^{(2)}$ conditioned on $M^{(2)} \cap T^{-1}M^{(3)}$, and (ii) for any $j_1, j_2 \geq 1$ we have

$$\lim_{i\to\infty} r_{i,j_1}^{(123)}/r_{i,j_1}^{(123)} = q_{j_1}^{(23)}/q_{j_2}^{(23)}$$

Theorem 15. *Under the conditions of the previous theorem we have*

(i) *If* $\lambda_+^{(1)} > \max\{\lambda_+^{(2)}, \lambda_+^{(3)}\}$, *then the measure* $\eta_+ = \lim T_*^{-n}\mu_+$ *coincides with* $\eta_+^{(1)}$.

(ii) *Let* $\lambda_+^{(1)} \leq \lambda_+^{(2)}$ *and* $\lambda_+^{(2)} > \lambda_+^{(3)}$. *Then* $\eta_+ = \eta_+^{(2)}$.

(iii) *If* $\lambda_+^{(3)} \geq \max\{\lambda_+^{(1)}, \lambda_+^{(2)}\}$ *then* $\eta_+ = \eta_+^{(3)}$.

In every case η_+ *is an ergodic* T*-invariant Gibbs measure with potential function* $g(x) = -\ln J^u(x)$ *and topological pressure* $P = \ln \lambda_+$. *It satisfies the equation (2.10).*

The last configuration, IV, can be reduced to III by eliminating the flow of mass from $M^{(1)}$ to $M^{(2)}$ together with the secondary flow from $M^{(1)}$ to $M^{(3)}$ via $M^{(2)}$. In the previous two theorems this forces $q_k^{(12)} = 0$ and $r_{m,n}^{(123)} = 0$ for all k, m, n. Then the results of those theorems apply to the configuration IV in the cases (i) and (iii). The case (ii) goes through under an additional assumption that $\lambda_+^{(1)} < \lambda_+^{(2)}$. The possibility $\lambda_+^{(1)} = \lambda_+^{(2)} > \lambda_+^{(3)}$ is treated separately in the following theorem.

Theorem 16. *Assume the configuration (IV), the mixing condition within every TT group of rectangles, and the absence of outgoing and transmitting rectangles in the system. Let* $\lambda_+^{(1)} = \lambda_+^{(2)} > \lambda_+^{(3)}$. *Then any regular conditionally invariant measure for* T *is a weighted sum* $\mu_+ = w_1\mu_{+,1} + w_2\mu_{+,2}$, *where*

$$\mu_{+,i} = \mu_+^{(i)} + \sum_{k=1}^{\infty} q_k^{(i3)}[\lambda_+^{(i)}]^{-k}\mu_k^{(i3)}$$

for $i = 1, 2$. *The measures* $\mu_{+,i}$ *are singular with respect to each other. The eigenvalue of any such* μ_+ *is* $\lambda_+^{(1)} = \lambda_+^{(2)}$.

For any smooth measure μ *on* M *positive on every open set the sequence* $T_+^n\mu$ *weakly converges, as* $n \to \infty$, *to some regular conditionally* *invariant measure μ_+, whose weights are determined by the initial distribution of μ between the TT components.*

The T-invariant measure $\eta_+ = \lim T_*^{-k}\mu_+$ is a weighted sum of $\eta_+^{(1)}$ and $\eta_+^{(2)}$. It is a Gibbs measure with potential function $g(x) = -\ln J^u(x)$ and topological pressure $P = \ln\lambda_+$. It satisfies the equation (2.10) and has two ergodic components.

8. General conclusions

Here we generalize (leaving out some technical details and exact proofs) the theorems obtained in the previous two sections to systems with arbitrary number of TT components.

First of all, if the system has an only dominating TT component (the one with the largest eigenvalue), then Rule 1 describes the properties of measures μ_+ and η_+.

If the system has more than one dominating TT components, then we subdivide them into essential and nonessential ones as follows. Any dominating component $M^{(i)}$ that has a one-way connection *to* another dominating component (possibly, via some transmitting rectangles and/or other TT components) is said to be *nonessential.* The remaining dominating components are *essential.*

Rule 2. The measures μ_+ and η_+ always exist. They are determined by essential dominating (ED) components only. If the system has just one ED component, the measures μ_+ and η_+ are unique and determined by that component according to the Rule 1, as if all the other TT components were not even dominating.

Rule 3. If the system has two or more ED components, the measures μ_+ and η_+ are not unique. Every ED component $M^{(i)}$ determines measures $\mu_{+,i}$ and $\eta_{+,i}$ according to the Rule 2. The measures $\mu_{+,i}$ are singular with respect to each other. The set of regular conditionally invariant measures μ_+ for T is the convex hull of the measures $\mu_{+,i}$. They have the same eigenvalue, which is the common eigenvalue of all ED components. The set of regular invariant measures η_+ for T is the convex hull of the measures $\eta_{+,i}$. Every η_+ is a Gibbs measure with potential function $g(x) = -\ln J^u(x)$ and topological pressure $P = \ln \lambda_+$, and it satisfies the equation (2.10). The number of its ergodic components equals the number of ED components in the system.

We do not prove Rules 2 and 3 in the general case, since they directly generalize our theorems proved in two previous sections. We also leave out detailed description of the measures $\mu_{+,i}$ that was provided in the previous sections. We could have given such a description along the lines developed in the case (i) of Theorems 10, 12 and 14, but it would involve unpleasantly heavy, though conceptually simple, calculations. So, we restricted ourselves to the detailed analysis of two and three TT groups.

Finally, let us emphasize that Rules 2 and 3 do not require the topological mixing condition within TT components. This condition only affects the way the iterations of smooth measures converge to μ_+, and the way the iterations of $T_*^{-n}\mu_+$ converge to η_+. If the mixing condition within ED components fails, then the Cesaro limit of $T_*^n\mu$, for any smooth measure μ, is a measure μ_+^0 equivalent to μ_+, cf. Sect. 4. Also, the Cesaro limit of $T_*^{-n}\mu_+$ (and the limit of $T_*^{-n}\mu_+^0$) is η_+.

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